

# On rainbow matchings for hypergraphs

Hongliang Lu \*

School of Mathematics and Statistics  
Xi'an Jiaotong University  
Xi'an, Shaanxi 710049, China

Xingxing Yu †

School of Mathematics  
Georgia Institute of Technology  
Atlanta, GA 30332, USA

## Abstract

For any positive integer  $m$ , let  $[m] := \{1, \dots, m\}$ . Let  $n, k, t$  be positive integers. Aharoni and Howard conjectured that if, for  $i \in [t]$ ,  $\mathcal{F}_i \subset [n]^k := \{(a_1, \dots, a_k) : a_j \in [n] \text{ for } j \in [k]\}$  and  $|\mathcal{F}_i| > (t-1)n^{k-1}$ , then there exist  $M \subseteq [n]^k$  such that  $|M| = t$  and  $|M \cap \mathcal{F}_i| = 1$  for  $i \in [t]$ . We show that this conjecture holds when  $n \geq 3(k-1)(t-1)$ .

Let  $n, t, k_1 \geq k_2 \geq \dots \geq k_t$  be positive integers. Huang, Loh and Sudakov asked for the maximum  $\prod_{i=1}^t |\mathcal{R}_i|$  over all  $\mathcal{R} = \{\mathcal{R}_1, \dots, \mathcal{R}_t\}$  such that each  $\mathcal{R}_i$  is a collection of  $k_i$ -subsets of  $[n]$  for which there does not exist a collection  $M$  of subsets of  $[n]$  such that  $|M| = t$  and  $|M \cap \mathcal{R}_i| = 1$  for  $i \in [t]$ . We show that for sufficiently large  $n$  with  $\sum_{i=1}^t k_i \leq n(1 - (4k \ln n/n)^{1/k})$ ,  $\prod_{i=1}^t |\mathcal{R}_i| \leq \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \prod_{i=3}^t \binom{n}{k_i}$ . This bound is tight.

## 1 Introduction

For a positive integer  $k$  and a set  $S$ , let  $[k] := \{1, \dots, k\}$  and  $\binom{S}{k} := \{T \subseteq S : |T| = k\}$ . A *hypergraph*  $H$  consists of a vertex set  $V(H)$  and an edge set  $E(H) \subset 2^{V(H)}$ . Thus, for any positive integer  $n$ , any subset of  $2^{[n]}$  forms a hypergraph with vertex set  $[n]$ .

Let  $k$  be a positive integer. A hypergraph  $H$  is *k-uniform* if  $E(H) \subseteq \binom{V(H)}{k}$ , and a  $k$ -uniform hypergraph is also called a *k-graph*. A  $k$ -graph  $H$  is *k-partite* if there exists a partition of  $V(H)$  into sets  $V_1, \dots, V_k$  (called *partition classes*) such that for any  $f \in E(H)$ ,  $|f \cap V_i| = 1$  for  $i \in [k]$ .

Let  $H$  be a hypergraph and  $T \subseteq V(H)$ . We write  $e(H) := |E(H)| = |H|$ . (Note that we often identify  $E(H)$  with  $H$ .) The *degree* of  $T$  in  $H$ , denoted by  $d_H(T)$ , is the number of edges of  $H$  containing  $T$ . For any integer  $l \geq 0$ , let  $\delta_l(H) := \min\{d_H(T) : T \in \binom{V(H)}{l}\}$ .

---

\*luhongliang@mail.xjtu.edu.cn; partially supported by the National Natural Science Foundation of China under grant No.11471257 and Fundamental Research Funds for the Central Universities

†yu@math.gatech.edu; partially supported by NSF grants DMS-1265564 and DMS-1600387

denote the minimum  $l$ -degree of  $H$ . Hence,  $\delta_0(H)$  is the number of edges in  $H$ . Note that  $\delta_1(H)$  is often called the minimum *vertex* degree of  $H$ . If  $H$  is a  $k$ -graph then  $\delta_{k-1}(H)$  is also known as the minimum *codegree* of  $H$ .

Let  $H$  be a  $k$ -partite  $k$ -graph, with partition classes  $V_1, \dots, V_k$ . We say that  $H$  is *balanced* if  $|V_i| = |V_j|$  for all  $i, j \in [k]$ . A set  $T \subseteq V(H)$  is said to be *legal* if  $|T \cap V_i| \leq 1$  for all  $i \in [k]$ . Thus, if  $T$  is not legal in  $H$  then  $d_H(T) = 0$ . So for integer  $l$  with  $0 \leq l \leq k-1$ , let  $\delta_l(H) := \min\{d_H(T) : T \in \binom{V(H)}{l} \text{ and } T \text{ is legal}\}$ .

A *matching* in a hypergraph  $H$  is a set of pairwise disjoint edges in  $H$ , and we use  $\nu(H)$  to denote the maximum size of a matching in  $H$ . A classical problem in extremal set theory is to determine  $\max|H|$  with  $\nu(H)$  fixed. Erdős [4] in 1965 made the following conjecture: For positive integers  $k, n, t$ , every  $k$ -graph  $H$  on  $n$  vertices with  $\nu(H) < t$  satisfies  $e(H) \leq \max\left\{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\right\}$ . This bound is tight because of the complete  $k$ -graph on  $kt-1$  vertices and the  $k$ -graph on  $n$  vertices in which every edge intersects a fixed set of  $t$  vertices.

There has been attempts to extend the above conjecture of Erdős to a family of hypergraphs. Let  $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  be a family of hypergraphs. A set of pairwise disjoint edges, one from each  $\mathcal{F}_i$ , is called a *rainbow matching* for  $\mathcal{F}$ . (In this case, we also say that  $\mathcal{F}$  or  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  *admits* a rainbow matching.) Huang, Loh and Sudakov [6] and, independently, Aharoni and Howard [2] made the following conjecture: Let  $t$  be a positive integer and  $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  such that, for  $i \in [t]$ ,  $\mathcal{F}_i \subseteq \binom{[n]}{k}$  and  $|\mathcal{F}_i| > \left\{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\right\}$ ; then  $\mathcal{F}$  admits a rainbow matching. Huang, Loh and Sudakov [6] showed that this conjecture holds for  $n > 3k^2t$ . Aharoni and Howard [2] also proposed the following  $k$ -partite version.

**Conjecture 1.1** *If  $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  such that, for  $i \in [t]$ ,  $\mathcal{F}_i$  is a  $k$ -partite  $k$ -graph in which each partition class has size  $n$ , then  $\mathcal{F}$  admits a rainbow matching.*

Aharoni and Howard [2] proved Conjecture 1.1 for  $t = 2$  or  $k \leq 3$ . Our first result implies that Conjecture 1.1 holds when  $n \geq 3(k-1)(t-1)$ .

**Theorem 1.2** *Let  $k, r, n, t$  be positive integers such that  $2 \leq r \leq k$  and  $n \geq 3(k-1)(t-1)$ , and let  $U_1, \dots, U_k$  be pairwise disjoint sets with  $|U_i| = n$  for  $i \in [k]$ . For each  $i \in [t]$ , let  $i_1, \dots, i_r \in [k]$  be pairwise distinct such that  $\mathcal{F}_i \subseteq U_{i_1} \times \dots \times U_{i_r}$  and  $|\mathcal{F}_i| > (t-1)n^{r-1}$ . Then  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  admits a rainbow matching.*

The famous Erdős-Ko-Rado theorem states that if  $k \leq n/2$  and  $H \subset \binom{[n]}{k}$  has more than  $\binom{n-1}{k-1}$  edges, then  $\nu(H) > 1$ . Pyber [8] gave a product-type generalization of the Erdős-Ko-Rado theorem, which was improved by Matsumoto and Tokushige [7] to the following: Let  $k_1, k_2, n$  be positive integers such that  $n \geq \max\{2k_1, 2k_2\}$ , and let  $H_i \subset \binom{[n]}{k_i}$  for  $i \in [2]$  such that  $e(H_1)e(H_2) > \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}$ ; then  $\{H_1, H_2\}$  admits a rainbow matching. Huang, Loh and Sudakov [6] asked the following more general question.

**Problem 1.3** *For positive integers  $n, t, k_1, \dots, k_t$ , what is the maximum  $\Pi_{i=1}^t |\mathcal{R}_i|$  among families  $\mathcal{R} = \{\mathcal{R}_1, \dots, \mathcal{R}_t\}$  such that  $\mathcal{R}_i \subseteq \binom{[n]}{k_i}$  for  $i \in [t]$  and  $\mathcal{R}$  admits no rainbow matching.*

Our second result in this paper provides an answer to Problem 1.3 when  $n$  is large.

**Theorem 1.4** *Let  $n, t, k_1, \dots, k_t$  be positive integers such that  $n$  is sufficiently large and  $\sum_{i=1}^t k_i \leq n(1 - (\frac{8k \ln n}{n})^{1/k})$ . Suppose  $k_1, k_2 \geq k_i$  for  $i = 3, \dots, t$ . Let  $\mathcal{F}_i \subset \binom{[n]}{k_i}$  for  $i \in [t]$ , such that*

$$|\mathcal{F}_1| |\mathcal{F}_2| > \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \quad \text{when } t = 2,$$

and

$$\prod_{i=1}^t |\mathcal{F}_i| > \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \prod_{i=3}^t \binom{n}{k_i} \quad \text{when } t \geq 3.$$

Then  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  admits a rainbow matching.

We remark that the bound in Theorem 1.4 is tight. Let  $\mathcal{F}_i = \{e : 1 \in e \in \binom{[n]}{k_i}\}$  for  $i \in [2]$  and let  $\mathcal{F}_i = \binom{[n]}{k_i}$  for  $i \in [t] - \{1, 2\}$ . Then

$$\prod_{i=1}^t |\mathcal{F}_i| = \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \prod_{i=3}^t \binom{n}{k_i}.$$

Clearly,  $\{\mathcal{F}_1, \mathcal{F}_2\}$  does not admit any rainbow matching. Hence,  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  does not admit any rainbow matching.

Our third result is a natural extension of Theorem 3.3 in [6] by Huang, Loh and Sudakov.

**Theorem 1.5** *Let  $n, t, k_1, \dots, k_t$  be positive integers such that  $n > 3k^2t$ , and let  $k = \max\{k_i : i \in [t]\}$ . For  $i \in [t]$ , let  $\mathcal{F}_i \subset \binom{[n]}{k_i}$  such that  $|\mathcal{F}_i| > \binom{n}{k_i} - \binom{n-t+1}{k_i}$ . Then  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  admits a rainbow matching.*

In view of Theorem 1.5, we ask the following

**Question 1.6** *Let  $k, n, t$  be positive integers and let  $\varepsilon$  be a constant such that  $0 < \varepsilon < 1$  and  $n \geq kt/(1 - \varepsilon)$ , and let  $\mathcal{R}_i \subset \binom{[n]}{k}$  for  $i \in [t]$  such that  $|\mathcal{R}_1| \leq |\mathcal{R}_2| \leq \dots \leq |\mathcal{R}_t|$ . Is it true that if for all  $r \in [t]$ ,*

$$\prod_{i=1}^r |\mathcal{R}_i| > \left( \binom{n}{k} - \binom{n-r+1}{k} \right)^r,$$

then  $\{\mathcal{R}_1, \dots, \mathcal{R}_t\}$  admits a rainbow matching?

## 2 Rainbow matchings

In this section we prove Theorem 1.2. First, we prove the following lemma, which will serve as basis for our inductive proof of Theorem 1.2.

**Lemma 2.1** *Let  $n > t > 0$  be integers, and let  $U_1, \dots, U_k$  be pairwise disjoint sets with  $|U_i| = n$  for  $i \in [k]$ . For each  $i \in [t]$ , let  $i_1, i_2 \in [k]$  be distinct and let  $F_i \subset U_{i_1} \times U_{i_2}$  such that  $e(F_i) > (t-1)n$  for  $i \in [t]$ . Then  $\{F_1, \dots, F_t\}$  admits a rainbow matching.*

**Proof.** Since  $e(F_1) > (t-1)n$ , there exists  $x_1 \in V(F_1)$  such that  $d_{F_1}(x_1) \geq t$ . Since  $e(F_2) > (t-1)n$ , there exists  $x_2 \in V(F_2)$  such that  $d_{F_2-x_1}(x_2) \geq t-1$ . Suppose that we have chosen  $x_{s-1}$ , where  $2 \leq s-1 \leq t-1$ , such that  $d_{F_{s-1}-\{x_1, \dots, x_{s-2}\}}(x_{s-1}) \geq t-(s-2)$ . Let  $X_{s-1} = \{x_1, \dots, x_{s-1}\}$ ,  $|X_{s-1} \cap U_{s_1}| = a$ , and  $|X_{s-1} \cap U_{s_2}| = b$ . Then  $a+b \leq |X_{s-1}| = s-1$ .

Without loss of generality, we may assume  $a \geq b$ . Then

$$\begin{aligned} e(F_s - X_{s-1}) &> (t-1)n - (an + bn - ab) \\ &= (t-1 - (a+b))n + ab \\ &\geq (t-s)n + ab. \end{aligned}$$

Hence,  $F_s - X_{s-1}$  contains a vertex  $x_s$  such that

$$d_{F_s - X_{s-1}}(x_s) > \frac{(t-s)n + ab}{n-a} > t-s.$$

Therefore, we obtain a sequence  $x_1, \dots, x_t$  of distinct elements of  $\bigcup_{i \in [k]} U_i$  such that  $d_{F_s - X_{s-1}}(x_s) \geq t - (s-1)$  for  $s \in [t]$ , where  $X_0 = \emptyset$ .

We now show that the desired rainbow matching exists by finding edges  $e_s \in F_s$  in the order  $s = t, \dots, 1$ . Since  $d_{F_t - X_{t-1}}(x_t) \geq 1$ , there exists  $e_t \in F_t$  such that  $e_t \cap X_{t-1} = \emptyset$  and  $x_t \in e_t$ . Suppose we have found pairwise disjoint edges  $e_t, \dots, e_{s+1}$  for some  $s \in [t-1]$ , such that, for  $s+1 \leq j \leq t$ ,  $e_j \in F_j$ ,  $e_j \cap X_{j-1} = \emptyset$ , and  $x_j \in e_j$ . Since  $F_s$  is bipartite,  $x_s$  is adjacent to at most one vertex of each  $e_j$ , for  $s+1 \leq j \leq t$ . Thus, since  $d_{F_s - X_{s-1}}(x_s) \geq t - (s-1)$ , there exists  $e_s \in F_s$  such that  $e_s \cap X_{s-1} = \emptyset$  and  $x_s \in e_s$ . Hence, by induction, there exist pairwise disjoint edges  $e_1, \dots, e_t$  which form a rainbow matching for  $\{F_1, \dots, F_t\}$ .  $\square$

**Proof of Theorem 1.2.** We apply induction on  $t+r$ . Clearly, the assertion holds for  $t=1$ . For  $r=2$ , the assertion follows from Lemma 2.1. Therefore, we may assume  $t \geq 2$  and  $r \geq 3$ , and that the assertion holds with smaller  $t+r$ .

Suppose for  $i \in [t]$ ,  $|\{x \in V(\mathcal{F}_i) : d_{\mathcal{F}_i}(x) > 2(t-1)n^{r-2}\}| \geq t$ . Then there exist pairwise distinct  $x_1, \dots, x_t$  such that, for  $i \in [t]$ ,  $x_i \in V(\mathcal{F}_i)$  and  $d_{\mathcal{F}_i}(x_i) > 2(t-1)n^{r-2}$ . Let  $X := \{x_1, \dots, x_t\}$  and for  $i \in [t]$ , let  $\mathcal{F}'_i := \{S : S \subset V(\mathcal{F}_i) - (X - x_i) \text{ and } S \cup \{x_i\} \in \mathcal{F}_i\}$ . Then, for  $i \in [t]$ ,

$$|\mathcal{F}'_i| = d_{\mathcal{F}_i - (X - x_i)}(x_i) > 2(t-1)n^{r-2} - (t-1)n^{r-2} = (t-1)n^{r-2}.$$

By induction hypothesis, let  $\{e_1, \dots, e_t\}$  be a rainbow matching for  $\{\mathcal{F}'_1, \dots, \mathcal{F}'_t\}$ , with  $e_i \in \mathcal{F}'_i$  for  $i \in [t]$ . Clearly,  $\{e_1 \cup \{x_1\}, \dots, e_t \cup \{x_t\}\}$  is a rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$ .

Hence, we may assume, without loss of generality, that  $|\{x \in V(\mathcal{F}_t) : d_{\mathcal{F}_t}(x) > 2(t-1)n^{r-2}\}| \leq t-1$ . By induction hypothesis, there exists a rainbow matching  $M'$  for  $\{\mathcal{F}_1, \dots, \mathcal{F}_{t-1}\}$ .

Suppose  $d_{\mathcal{F}_t}(x) \leq (t-1)(r-1)n^{r-2}$  for all  $x \in V(\mathcal{F}_t)$ . Then, since  $r \geq 3$  and  $|\{x \in V(\mathcal{F}_t) : d_{\mathcal{F}_t}(x) > 2(t-1)n^{r-2}\}| \leq t-1$ , the number of edges in  $\mathcal{F}_t$  intersecting  $V(M')$  is less than

$$\begin{aligned} &(t-1)((t-1)(r-1)n^{r-2}) + ((t-1)r - (t-1))(2(t-1)n^{r-2}) \\ &= (t-1)^2(3r-3)n^{r-2} \\ &\leq (t-1)n^{r-1}, \quad \text{since } n \geq (3k-3)(t-1) \text{ and } k \geq r \geq 3. \end{aligned}$$

So there exists  $e \in \mathcal{F}_t - V(M')$ . Hence  $M' \cup \{e\}$  is rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$ .

Therefore, we may assume that there exists  $x \in V(\mathcal{F}_t)$  such that  $d_{\mathcal{F}_t}(x) > (t-1)(r-1)n^{r-2}$ . For  $i \in [t-1]$ , let  $\mathcal{F}_i'' = \mathcal{F}_i - \{e \in \mathcal{F}_i : x \in e\}$ . Then

$$|\mathcal{F}_i''| > (t-1)n^{r-1} - n^{r-1} = (t-2)n^{r-1}.$$

Hence, by induction hypothesis, there exists a rainbow matching  $M$  for  $\{\mathcal{F}_1'', \dots, \mathcal{F}_{t-1}''\}$ . Since  $d_{\mathcal{F}_i''}(x) = 0$  for  $i \in [t-1]$ ,  $x \notin V(M)$ .

Since the number of edges in  $\mathcal{F}_t$  containing  $x$  and intersecting  $V(M)$  is at most  $(t-1)(r-1)n^{r-2} < d_{\mathcal{F}_t}(x)$ , there exists  $e \in \mathcal{F}_t - V(M)$ . So  $M \cup \{e\}$  gives the desired rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$ .  $\square$

We now prove Conjecture 1.1 for the case when  $t = n$ .

**Proposition 2.2** *Let  $k, n, t$  be positive integers with  $t \leq n$ , and for  $i \in [k]$ , let  $W_i = \{jk + i : j \in [n-1] \cup \{0\}\}$ . For  $i \in [n]$ , let  $\mathcal{F}_i \subset W_1 \times \dots \times W_k$  such that  $|\mathcal{F}_i| > (t-1)n^{k-1}$ . Then there exist pairwise distinct  $i_1, \dots, i_t$  from  $[k]$  such that  $\{\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_t}\}$  admits a rainbow matching.*

**Proof.** Consider a permutation  $\pi$  of  $[kn]$ , taken uniformly at random from all permutations  $\pi$  of  $[kn]$  with the property that  $\pi(W_i) = W_i$  for all  $i \in [k]$ . For  $i \in [n]$ , let  $X_i = 1$  if  $\{\pi((i-1)k+1), \pi((i-1)k+2), \dots, \pi(ik)\} \in \mathcal{F}_i$ , and let  $X_i = 0$  otherwise. Then

$$\mathbb{P}(X_i = 1) = \frac{|\mathcal{F}_i|}{n^k} > \frac{(t-1)}{n}.$$

Hence

$$\mathbb{E} \left( \sum_{i=1}^n X_i \right) = n\mathbb{E}(X_i = 1) > t-1.$$

Therefore, there exist pairwise distinct  $i_1, \dots, i_t$  from  $[k]$  such that  $X_{i_j} = 1$  for  $j \in [t]$ . Hence,  $\{\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_t}\}$  admits a rainbow matching.  $\square$

Setting  $t = n$  in Proposition 2.2, we obtain the following result on perfect matchings.

**Corollary 2.3** *Let  $n, k$  be positive integers, and let  $\mathcal{F}_i \subset [n]^k$  for  $i \in [n]$ . If  $|\mathcal{F}_i| > (n-1)n^{k-1}$  for  $i \in [n]$ , then  $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$  admits a rainbow matching.*

Setting  $\mathcal{F}_1 = \dots = \mathcal{F}_n$  in Proposition 2.2, we obtain the following well-known result .

**Corollary 2.4** *Let  $n, k$  be positive integers, and let  $H$  be a  $k$ -partite  $k$ -graph with  $V(H) = [n]^k$ . If  $e(H) > (t-1)n^{k-1}$ , then  $\nu(H) \geq t$ .*

### 3 Product type conditions

In this section we prove Theorem 1.4. First, we state a result of Matsumoto and Tokushige [7].

**Lemma 3.1** Let  $k_1, k_2, n$  be positive integers such that  $n \geq \max\{2k_1, 2k_2\}$ , and let  $H_i \subset \binom{[n]}{k_i}$ ,  $i \in [2]$ , such that  $e(H_1)e(H_2) > \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}$ . Then  $\{H_1, H_2\}$  admits a rainbow matching.

We use Lemma 3.1 as induction basis to prove the next result.

**Lemma 3.2** Let  $k, t, n$  be integers such that  $t \geq 2$ ,  $k_1 \geq k_2 \geq \dots \geq k_t \geq 2$ , and  $n \geq 9k_1^5 t/k_2$ . Let  $\mathcal{F}_i \subset \binom{[n]}{k_i}$  for  $i \in [t]$ , such that

$$|\mathcal{F}_1| |\mathcal{F}_2| > \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \quad \text{when } t=2,$$

and

$$\prod_{i=1}^t |\mathcal{F}_i| > \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \prod_{i=3}^t \binom{n}{k_i} \quad \text{when } t \geq 3.$$

Then  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  admits a rainbow matching.

**Proof.** If  $t = 2$  then the assertion follows from Lemma 3.1. Thus, we may assume that  $t \geq 3$  and the assertion holds with fewer than  $t$  families. Let  $s \in [t]$  such that

$$\frac{|\mathcal{F}_s|}{\binom{n}{k_s}} = \max \left\{ \frac{|\mathcal{F}_i|}{\binom{n}{k_i}} : i \in [t] \right\}.$$

Since  $|\mathcal{F}_s| \leq \binom{n}{k_s}$ , if  $s \notin [2]$  then

$$\prod_{i \in [t] - \{s\}} |\mathcal{F}_i| > \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \prod_{i \in [t] - \{1,2,s\}} \binom{n}{k_{\pi(i)}},$$

and if  $s \in [2]$  then

$$\begin{aligned} \prod_{i \in [t] - \{s\}} |\mathcal{F}_i| &> \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \prod_{i=3}^t \binom{n}{k_i} / \binom{n}{k_s} \\ &= \frac{k_s}{n} \binom{n-1}{k_{3-s}-1} \prod_{i=3}^t \binom{n}{k_i} \\ &= \frac{k_s}{k_3} \binom{n-1}{k_{3-s}-1} \binom{n-1}{k_3-1} \prod_{i=4}^t \binom{n}{k_i} \\ &\geq \binom{n-1}{k_{3-s}-1} \binom{n-1}{k_3-1} \prod_{i=4}^t \binom{n}{k_i} \quad (\text{since } k_1 \geq k_2 \geq k_3). \end{aligned}$$

By induction hypothesis,  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\} - \{\mathcal{F}_s\}$  admits a rainbow matching, say  $M$ . Note that the number of edges in  $\mathcal{F}_s$  intersecting  $V(M)$  is at most

$$\left( \sum_{i \in [t] - \{s\}} k_i \right) \binom{n-1}{k_s-1}.$$

Hence, if  $|\mathcal{F}_s| > (\sum_{i \in [t] - \{s\}} k_i) \binom{n-1}{k_s-1}$  then there exists  $e \in \mathcal{F}_s$  disjoint from  $V(M)$ . Thus  $M \cup \{e\}$  is the desired rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$ .

So we may assume that  $|\mathcal{F}_s| \leq (\sum_{i \in [t] - \{s\}} k_i) \binom{n-1}{k_s-1}$ . Then, since  $k_i \leq k_1$  for  $i \in [t]$ ,

$$|\mathcal{F}_s| \leq k_1(t-1) \frac{k_s}{n} \binom{n}{k_s} \leq \frac{k_1^2(t-1)}{n} \binom{n}{k_s}.$$

Therefore,

$$\prod_{i=1}^t \frac{|\mathcal{F}_i|}{\binom{n}{k_i}} \leq \left( \frac{|\mathcal{F}_s|}{\binom{n}{k_s}} \right)^t \leq \left( \frac{k_1^2(t-1)}{n} \right)^t < \left( \frac{k_1^2 t}{n} \right)^t.$$

On the other hand, by assumption of this lemma,

$$\prod_{i=1}^t \frac{|\mathcal{F}_i|}{\binom{n}{k_i}} > \frac{\binom{n-1}{k_1-1} \binom{n-1}{k_2-1}}{\binom{n}{k_1} \binom{n}{k_2}} = \frac{k_1 k_2}{n^2}.$$

Hence,

$$\left( \frac{k_1^2 t}{n} \right)^t > \frac{k_1 k_2}{n^2}.$$

Thus,

$$f(t) := t(\ln k_1^2 + \ln t - \ln n) - \ln(k_1 k_2) + 2 \ln n > 0.$$

However, the derivative  $f'(t) = \ln(k_1^2) + \ln t - \ln n + 1 < 0$ , since  $n \geq 9k_1^5 t / k_2 > 3k_1^2 t$ . Thus  $f(t)$  is a decreasing function. Hence, since  $3 \leq t \leq k_2 n / (9k_1^5)$  and  $n \geq 9k_1^5 t / k_2$ , we have

$$f(t) \leq f(3) = 3(\ln k_1^2 + \ln 3 - \ln n) - \ln(k_1 k_2) + 2 \ln n < 0,$$

a contradiction. □

We need another lemma.

**Lemma 3.3** *Let  $t, n, k_1, \dots, k_t$  be positive integers and let  $\varepsilon > 0$  be a small constant such that  $k_1 \geq k_2 \geq \dots \geq k_t$ ,  $\sum_{i=1}^t k_i \leq n(1 - \varepsilon)$ , and  $n$  is sufficiently large. For  $i \in [t]$ , let  $\mathcal{F}_i \subseteq \binom{[n]}{k_i}$  such that*

$$|\mathcal{F}_1| |\mathcal{F}_2| > \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \quad \text{when } t=2,$$

and

$$\prod_{i=1}^t |\mathcal{F}_i| > \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \prod_{i=3}^t \binom{n}{k_i} \quad \text{when } t \geq 3.$$

Then  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  admits a rainbow matching.

**Proof.** By Lemma 3.2, we may assume that  $t \geq 9k_1^5 n/k_2$ . Since  $n$  is sufficiently large, we have  $t \geq 3$ . Let  $s \in [t]$  such that

$$\frac{|\mathcal{F}_s|}{\binom{n}{k_s}} = \max \left\{ \frac{|\mathcal{F}_i|}{\binom{n}{k_i}} : i \in [t] \right\}.$$

As induction hypothesis, we may assume that  $M$  is a rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\} - \{\mathcal{F}_s\}$ . Then  $|V(M)| = \sum_{i \in [t] - \{s\}} k_i$ . Again since  $n$  is sufficiently large and  $\sum_{i \in [t]} k_i \leq n(1 - \varepsilon)$ , the number of edges in  $\mathcal{F}_s$  intersecting  $V(M)$  is at most

$$\binom{n}{k_s} - \binom{n - \sum_{i \in [t] - \{s\}} k_i}{k_s} \leq \binom{n}{k_s} - \binom{n\varepsilon}{k_s} < \binom{n}{k_s} \left(1 - \frac{1}{2}\varepsilon^{k_s}\right).$$

If  $|\mathcal{F}_s| > \binom{n}{k_s} - \binom{n - \sum_{i \in [t] - \{s\}} k_i}{k_s}$  then there exists  $e \in \mathcal{F}_s$  disjoint from  $V(M)$ ; so  $M \cup \{e\}$  is the desired rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$ .

So assume  $|\mathcal{F}_s| \leq \binom{n}{k_s} - \binom{n - \sum_{i \in [t] - \{s\}} k_i}{k_s}$ . By assumption of this lemma,

$$\prod_{i=1}^t \frac{|\mathcal{F}_i|}{\binom{n}{k_i}} > \frac{\binom{n-1}{k_1} \binom{n-1}{k_2}}{\binom{n}{k_1} \binom{n}{k_2}} = \frac{k_1 k_2}{n^2}.$$

However, since  $t \geq 9k_1^5 n/k_2 \geq 9k_1^4 n$ ,

$$\frac{|\mathcal{F}_s|}{\binom{n}{k_s}} \geq \left( \prod_{i=1}^t \frac{|\mathcal{F}_i|}{\binom{n}{k_i}} \right)^{1/t} > \left( \frac{k_1 k_2}{n^2} \right)^{1/(9k_1^4 n)}.$$

Since

$$\lim_{n \rightarrow +\infty} \left( \frac{k_1 k_2}{n^2} \right)^{1/(9k_1^4 n)} = 1$$

and  $n$  is sufficiently large,

$$\frac{|\mathcal{F}_s|}{\binom{n}{k_s}} > 1 - \frac{1}{2}\varepsilon^{k_s}.$$

This is a contradiction. □

We also need the following inequality when  $\sum_{i \in [t]} k_i \geq n(1 - \varepsilon)$ .

**Lemma 3.4** *Let  $k, t, n$  be positive integers and  $\varepsilon > 0$  be a small constant, such that  $n$  is sufficiently large,  $k_1 \geq k_2 \geq \dots \geq k_t \geq 2$ , and*

$$n(1 - \varepsilon) \leq \sum_{i=1}^t k_i \leq n - n \left( \frac{8k_1 \ln n}{n} \right)^{1/k_1}.$$

*Then*

$$\left( \frac{k_1 k_2}{n^2} \right)^{1/t} > 1 - \frac{1}{2} \left( 1 - \frac{\sum_{i=1}^t k_i}{n} \right)^{k_1}.$$



**Proof.** Let  $m := n - \sum_{i=1}^t k_i$ . Then, by the assumption of this lemma,

$$n \left( \frac{8k_1 \ln n}{n} \right)^{1/k_1} \leq m \leq n\varepsilon.$$

Moreover,  $m = n - \sum_{i=1}^t k_i \geq n - tk_1$ ; so  $n\varepsilon \geq n - tk_1$  and, hence,  $t \geq n(1 - \varepsilon)/k_1$ . Since  $n$  is large, we may assume  $n^2 > k_1 k_2$ . Hence

$$\frac{1}{t} \ln \left( \frac{k_1 k_2}{n^2} \right) \geq \frac{k_1}{n(1 - \varepsilon)} \ln \frac{k_1 k_2}{n^2} > \frac{4k_1}{3n} \ln \frac{k_1 k_2}{n^2},$$

where the last inequality holds as  $\varepsilon$  is small (say  $\varepsilon < 1/4$ ).

Note that the assertion of this lemma is equivalent to

$$\left( \frac{k_1 k_2}{n^2} \right)^{1/t} > 1 - \frac{1}{2} \left( \frac{m}{n} \right)^{k_1},$$

which holds iff

$$\frac{1}{t} \ln \left( \frac{k_1 k_2}{n^2} \right) > \ln \left( 1 - \frac{1}{2} \left( \frac{m}{n} \right)^{k_1} \right).$$

Thus, since  $n$  is large, it suffices to show

$$\frac{4k_1}{3n} \ln \frac{k_1 k_2}{n^2} > -\frac{1}{3} \left( \frac{m}{n} \right)^{k_1}.$$

However, this follows from a straightforward calculation, using  $m \geq n \left( \frac{8k_1 \ln n}{n} \right)^{1/k_1}$ .  $\square$

**Proof of Theorem 1.4.** By Lemma 3.1, we may assume  $t \geq 3$ . By Lemma 3.3 and by assumption, we may assume that

$$n(1 - \varepsilon) \leq \sum_{i=1}^t k_i \leq n - n \left( \frac{8k_1 \ln n}{n} \right)^{1/k_1}.$$

Let  $s \in [t]$  such that

$$\frac{|\mathcal{F}_s|}{\binom{n}{k_s}} = \max \left\{ \frac{|\mathcal{F}_i|}{\binom{n}{k_i}} : i \in [t] \right\}.$$

As induction hypothesis, assume that  $M$  is a rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\} - \{\mathcal{F}_s\}$ . Then  $|V(M)| = \sum_{i \in [t] - \{s\}} k_i$  and the number of edges in  $\mathcal{F}_s$  intersecting  $V(M)$  is at most

$$\binom{n}{k_s} - \binom{n - \sum_{i \in [t] - \{s\}} k_i}{k_s}.$$

Note that

$$\frac{|\mathcal{F}_s|}{\binom{n}{k_s}} \geq \left( \prod_{i=1}^t \frac{|\mathcal{F}_i|}{\binom{n}{k_i}} \right)^{1/t} > \frac{\binom{n-1}{k_1-1} \binom{n-1}{k_2-1}}{\binom{n}{k_1} \binom{n}{k_2}} = \left( \frac{k_1 k_2}{n^2} \right)^{1/t}.$$

Hence, by Lemma 3.4 (and since  $n$  is large),

$$\frac{|\mathcal{F}_s|}{\binom{n}{k_s}} > 1 - \frac{1}{2} \left( 1 - \frac{\sum_{i=1}^t k_i}{n} \right)^{k_1} > 1 - \frac{\binom{n - \sum_{i=1}^t k_i}{k_s}}{\binom{n}{k_s}}.$$

Therefore,

$$|\mathcal{F}_s| > \binom{n}{k_s} - \binom{n - \sum_{i=1}^t k_i}{k_s}.$$

So there exists  $e \in \mathcal{F}_s$  such that  $e \cap V(M) = \emptyset$ . Now  $M \cup \{e\}$  is a rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$ .  $\square$

## 4 Proof of Theorem 1.5

Suppose the assertion of Theorem 1.5 is false. We choose a counterexample so that  $t$  is minimum and, subject to this,  $\sum_{i=1}^t k_i$  is minimum. Clearly,  $t \geq 2$ .

We claim that  $k_i \geq 2$  for  $i \in [t]$  and, for  $i \in [t]$  and  $v \in [n]$ ,  $d_{\mathcal{F}_i}(v) \leq k(t-1)\binom{n-2}{k_i-2}$ . For, suppose there exist  $i \in [t]$  such that  $k_i = 1$  or  $d_{\mathcal{F}_i}(v) > k(t-1)\binom{n-2}{k_i-2}$  for some  $v \in \mathcal{F}_i$ . Note that, for  $j \in [t] - \{i\}$ ,

$$|\mathcal{F}_j - v| > \binom{n}{k_j} - \binom{n-t+1}{k_j} - \binom{n-1}{k_j-1} = \binom{n-1}{k_j} - \binom{n-t+1}{k_j}.$$

Hence,  $\{\mathcal{F}_1 - v, \dots, \mathcal{F}_t - v\} - \{\mathcal{F}_i - v\}$  admits a rainbow matching, say  $M$ . The number of edges in  $\mathcal{F}_i$  containing  $v$  and intersecting  $V(M)$  is at most

$$k(t-1)\binom{n-2}{k_i-2}.$$

So there exists  $e \in \mathcal{F}_i$  such that  $v \in e$  and  $e \cap V(M) = \emptyset$ . Therefore,  $M \cup \{e\}$  is a rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$ , a contradiction.

Suppose for each  $i \in [t]$ ,

$$\left| \left\{ v \in [n] : d_{\mathcal{F}_i}(v) > 2(t-1)\binom{n-2}{k_i-2} \right\} \right| \geq t.$$

Let  $v_1, \dots, v_t \in [n]$  be pairwise distinct such that  $d_{\mathcal{F}_i}(v_i) > 2(t-1)\binom{n-2}{k_i-2}$  for  $i \in [t]$ . Let  $S_i = \{v_j : j \in [t] - \{i\}\}$ ,  $i \in [t]$ . Then

$$d_{\mathcal{F}_i - S_i}(v_i) > 2(t-1)\binom{n-2}{k_i-2} - (t-1)\binom{n-2}{k_i-2} = (t-1)\binom{n-2}{k_i-2}.$$

For  $i \in [t]$ , let

$$\mathcal{F}'_i = \left\{ S : S \in \binom{[n] - \{v_1, \dots, v_t\}}{k_i-1} \text{ and } S \cup \{v_i\} \in \mathcal{F}_i - S_i \right\}.$$

Then

$$|\mathcal{F}'_i| = d_{\mathcal{F}_i - S_i}(v_i) > (t-1)\binom{n-2}{k_i-2}.$$

So  $\{\mathcal{F}'_1, \dots, \mathcal{F}'_t\}$  admits a rainbow matching, say  $\{e_1, \dots, e_t\}$  with  $e_i \in \mathcal{F}'_i$  for  $i \in [t]$ . Now  $\{e_1 \cup \{v_1\}, \dots, e_t \cup \{v_t\}\}$  is a rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$ , a contradiction.

Thus, without loss of generality, we may assume that

$$\left| \left\{ v \in [n] : d_{\mathcal{F}_t}(v) > 2(t-1) \binom{n-2}{k_t-2} \right\} \right| < t.$$

Let  $M$  be a rainbow matching for  $\{\mathcal{F}_1, \dots, \mathcal{F}_{t-1}\}$ . Since  $\{\mathcal{F}_1, \dots, \mathcal{F}_t\}$  admits no rainbow matching, every edge of  $\mathcal{F}_t$  must intersect  $V(M)$ . Hence,  $|\mathcal{F}_t|$  is at most

$$(k(t-1) - (t-1)) \binom{n-2}{k_t-2} + (t-1) \binom{k(t-1)}{k_t-2} = (3k-2)(t-1)^2 \binom{n-2}{k_t-2}.$$

On the other hand,

$$\begin{aligned} |\mathcal{F}_t| &> \binom{n}{k_t} - \binom{n-t+1}{k_t} \\ &= \binom{n}{k_t} - \binom{n}{k_t} \frac{(n-t+1) \cdots (n-t-k_t+2)}{n(n-1) \cdots (n-k_t+1)} \\ &> \binom{n}{k_t} \left( 1 - \left( 1 - \frac{t-1}{n} \right)^{k_t} \right) \\ &> \binom{n}{k_t} \left( \frac{k_t(t-1)}{n} - \frac{k_t^2(t-1)^2}{2n^2} \right) \\ &> \binom{n}{k_t} \frac{k_t(t-1)}{n} \left( 1 - \frac{1}{6k} \right) \quad (\text{since } n \geq 3k^2t \text{ and } k = \max\{k_i : i \in [t]\}) \\ &= \binom{n-2}{k_t-2} \frac{(n-1)(t-1)}{k_t-1} \left( 1 - \frac{1}{6k} \right) \\ &> \binom{n-2}{k_t-2} \frac{n(t-1)}{k_t} \left( 1 - \frac{1}{6k} \right), \end{aligned}$$

Therefore,

$$(3k-2)(t-1)^2 \binom{n-2}{k_t-2} > \binom{n-2}{k_t-2} \frac{n(t-1)}{k_t} \left( 1 - \frac{1}{6k} \right),$$

which implies

$$n < k_t(3k-2)(t-1) \frac{6k}{6k-1} < 3k^2t,$$

a contradiction.  $\square$

## References

- [1] R. Aharoni and E. Berger, Rainbow matchings in  $r$ -partite  $r$ -graphs, *Electronic Journal of Combinatorics*, **16** (2009), #R119.
- [2] R. Aharoni and D. Howard, Size conditions for the existence of rainbow matching, Preprint.

- [3] B. Bollobás, D.E. Daykin and P. Erdős, Sets of independent edges of a hypergraphs, *Quart. J. Math. Oxford Ser.*, **27** (1976), 25–32.
- [4] P. Erdős, A problem on independent  $r$ -tuples, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **8** (1965), 93–95.
- [5] P. Frankl, Improved bounds for Erdős matching conjecture, *J. Combin. Theory Ser. A*, **120** (2013), 1068–1072.
- [6] H. Huang, P. Loh and B. Sudakov, The size of a hypergraph and its matching number, *Combinatorics, Probability and Computing*, **21** (2012), 442–450.
- [7] M. Matsumoto and N. Tokushige, The exact bound in the Erdős-Ko-Rado theorem for cross-intersecting families, *J. Combin. Theory Ser. A*, **52** (1989), 90–97.
- [8] L. Pyber, A new generalization of the Erdős-Ko-Rado theorem, *J. Combin. Theory Ser. A*, **43** (1986), 85–90.